In this section, we introduce the Feynman parameterization, Lee-Pomeransky and Baikov representations of Feynman loop integrals.

For loop-level amplitudes, we have the following integrals

I.   FEYNMAN PARAMETRIZATION

We consider the Feynman parameterization in $(+-\ldots-)$ metric. Here we review the Feynman parametrization in a systematic way.

The integral under consideration is,

$$G[\alpha_1,\ldots\alpha_n] = \int \prod_{i=1}^{L} \frac{d^D l_i}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1}\ldots D_n^{\alpha_n}}\,. \tag{1}$$

The basic identity for Feynman parameterization is (with $\alpha_j > 0$),

$$\frac{1}{D_1^{\alpha_1}\ldots D_n^{\alpha_n}} = \frac{\Gamma(|\boldsymbol{\alpha}|)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^{n} dz_i \delta(1-\sum_j z_j) \frac{z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1}}{(z_1 D_1 + \ldots + z_n D_n)^{|\boldsymbol{\alpha}|}} \tag{2}$$

Here $|\boldsymbol{\alpha}|$ is the sum of α's. Define $l = (l_1,\ldots,l_L)^T$ as an L-component column vector. The denominator of (2) is a quadratic function of l,

$$z_1 D_1 + \ldots + z_n D_n \equiv l^T A l + 2b^T l + c\,, \tag{3}$$

where A is an $L \times L$ matrix, b is an L-component column vector and c is a scalar. A, b and c are l-independent. Since A is symmetric, A can be diagonalized as $A = O^T J O$ where J is diagonal. Consider the change of loop momenta, $l = O^T \tilde{l} - A^{-1} b$. Then

$$z_1 D_1 + \ldots + z_n D_n = \tilde{l}^T J \tilde{l} + c - b^T A^{-1} b\,. \tag{4}$$

Using Wick's rotation $\tilde{l}_i^0 = i\tilde{l}_{i,E}^0$ and $(\tilde{l}_i)^2 = -(\tilde{l}_{i,E})^2$, the integration contour is along the real axis of $\tilde{l}_{i,E}^0$.

$$I[\alpha_1,\ldots\alpha_n] = \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(|\boldsymbol{\alpha}|)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^{n} dz_i\ \delta(1-\sum_j z_j) z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1}$$

$$\times \int \prod_{i=1}^{L} \frac{d^D \tilde{l}_{i,E}}{\pi^{D/2}} \frac{1}{(\tilde{l}_E^T J \tilde{l}_E - c + b^T A^{-1} b - i\eta)^{|\boldsymbol{\alpha}|}} \tag{5}$$

$$= \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(|\boldsymbol{\alpha}| - \frac{DL}{2})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 \prod_{i=1}^{n} dz_i\ \delta(1-\sum_j z_j) z_1^{\alpha_1-1}\ldots z_n^{\alpha_n-1} \frac{F^{LD/2-|\boldsymbol{\alpha}|}}{U^{(L+1)D/2-|\boldsymbol{\alpha}|}} \tag{6}$$

where,

$$U = \det A\,, \quad F = -c \det A + b^T A^{\mathrm{adj}} b - i\eta \det A\,. \tag{7}$$

where $A^{\text{adj}} = (\det A)A^{-1}$ is the adjugate matrix of A. U and F are homogenous polynomials in z with the degree L and $L + 1$ respectively.

Note that U is a positive semidefinite polynomial but F may not not be positive semidefinite. If there is a kinematic region for which F is positive semidefinite, we call this region *Euclidean*. Note that for nonplanar loop integrals, the Euclidean region may not exist.

For the Euclidean region, both U and F are positive semidefinite. The integrand $F^{LD/2-|\alpha|}/U^{(L+1)D/2-|\alpha|}$ is well defined, and the integral must be *real*. For a region where F is not positive semidefinite, we need to include the infinitesimal η and the integral would be *complex*.

- Example: massless bubble integral. This is the simplest loop integral beyond the one-loop tadpole integral. The propagators are

$$D_1 = l_1^2, \quad D_2 = (l_1 - p)^2 . \tag{8}$$

The kinematics is $p^2 = s$.

$$U = z_1 + z_2 \tag{9}$$

$$F = -sz_1 z_2 \tag{10}$$

Note that $s < 0$ is the Euclidean region and the integral is real. $s > 0$ is the *physical* region and the integral is complex.

For this simple integral, the direction Feynman parametrization gives the analytical result

$$G[\alpha_1, \alpha_2] = \frac{(-1)^{\alpha_1 + \alpha_2} \Gamma\left(\frac{d}{2} - \alpha_1\right) \Gamma\left(\frac{d}{2} - \alpha_2\right) \Gamma\left(-\frac{d}{2} + \alpha_1 + \alpha_2\right) (-s)^{\frac{1}{2}(d - 2(\alpha_1 + \alpha_2))}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(d - \alpha_1 - \alpha_2)} . \tag{11}$$

The apparently simplest integral in this sector is $(D = 4 - 2\epsilon)$,

$$G[1, 1] = \frac{(-s)^{-\epsilon} \Gamma(1 - \epsilon)^2 \Gamma(\epsilon)}{\Gamma(2 - 2\epsilon)} \tag{12}$$

$$= \frac{1}{\epsilon} + (-\log(-s) - \gamma + 2)$$

$$+ \epsilon \left(\frac{2}{3} \left(-6\log^2(-s) - 12\gamma \log(-s) + 24\log(-s) + \pi^2 - 6\gamma^2 + 24\gamma - 48 \right) \right.$$

$$\left. + \frac{3}{4} \left(6\log^2(-s) + 12\gamma \log(-s) - 24\log(-s) - \pi^2 + 6\gamma^2 - 24\gamma + 48 \right) \right) +$$

$$\epsilon^2 \left(-\frac{1}{6}\log^3(-s) - \frac{1}{2}\gamma \log^2(-s) + \log^2(-s) + \frac{1}{12}\pi^2 \log(-s) - \frac{1}{2}\gamma^2 \log(-s) + 2\gamma \log(-s) \right.$$

$$\left. -4\log(-s) - \frac{7\zeta(3)}{3} + \frac{\gamma \pi^2}{12} - \frac{\pi^2}{6} - \frac{\gamma^3}{6} + \gamma^2 - 4\gamma + 8 \right) + O\left(\epsilon^3\right) \tag{13}$$

where γ is the Euler gamma constant. The Riemann Zeta function is defined as,

$$\zeta(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{14}$$

We may simplify the expansion by

$$e^{\epsilon\gamma}(-s)^{\epsilon} G[1,1] = \frac{1}{\epsilon} + 2 + \left(4 - \frac{\pi^2}{12}\right)\epsilon + \left(-\frac{7\zeta(3)}{3} + 8 - \frac{\pi^2}{6}\right)\epsilon^2 + \left(-\frac{14\zeta(3)}{3} + 16 - \frac{\pi^2}{3} - \frac{47\pi^4}{1440}\right)\epsilon^3 + O$$
$$\tag{15}$$

The overall transformation $e^{\epsilon\gamma}$ removed all Euler constants. Since this is a one-scale integral, $(-s)^{\epsilon}$ factor removed the s dependence. The rest contains irrational numbers π, $\zeta(3)$ (and more zeta values in the high order expansion).

However, the "simplest" integral is not $G[1,1]$ but the magic *uniformly transcendental* (UT) integral ([1]),

$$e^{\epsilon\gamma}(-s)^{\epsilon} s G[1,2] = -\frac{1}{\epsilon} + \frac{\pi^2\epsilon}{12} + \frac{7\zeta(3)\epsilon^2}{3} + \frac{47\pi^4\epsilon^3}{1440} + O\left(\epsilon^4\right). \tag{16}$$

Note that the ϵ^n order of this expression has the transcendental degree $n+1$. This expression is much more concise than (15).

So why is $G[1,1]$ complicated? The relation between the two integrals is that

$$G[1,1] = \frac{s}{-1 + 2\epsilon} G[1,2]. \tag{17}$$

The factor $-1 + 2\epsilon$ messed up the expression for $G[1,1]$.

The main breakthrough of modern loop computation is to guess UT integrals and then to evaluate them by the canonical differential equation [1].

- Massless planar double box. The propagators are

$$l_1^2, \quad (l_1 - k_1)^2, \quad (l_1 - k_1 - k_2)^2, \quad (l_2 + k_1 + k_2)^2, \quad (l_2 - k_4)^2, \quad l_2^2, \quad (l_1 + l_2)^2 \tag{18}$$

The polynomials are

$$U = z_1 z_4 + z_2 z_4 + z_3 z_4 + z_7 z_4 + z_1 z_5 + z_2 z_5 + z_3 z_5 + z_1 z_6 + z_2 z_6 + z_3 z_6 + z_1 z_7$$
$$+ z_2 z_7 + z_3 z_7 + z_5 z_7 + z_6 z_7$$
$$F = -s z_1 z_3 z_4 - s z_1 z_6 z_4 - s z_2 z_6 z_4 - s z_3 z_6 z_4 - s z_1 z_7 z_4 - s z_6 z_7 z_4 - s z_1 z_3 z_5$$
$$- s z_1 z_3 z_6 - s z_1 z_3 z_7 - s z_3 z_6 z_7 - t z_2 z_5 z_7 \tag{19}$$

Note that $s < 0$, $t < 0$ is the Euclidean region and the integral is real. Usually, we call $s > 0$, $t < 0$ the "physical" region.

- Massless nonplanar crossed box. The propagators are

$$l_1^2, (l_1 - k_1)^2, (l_1 - k_1 - k_2)^2, (l_2)^2, (l_2 - k_4)^2, (l_1 + l_2)^2, (l_1 + l_2 - k_1 - k_2 - k_4)^2 \quad (20)$$

The polynomials are,

$$U = z_1 z_4 + z_2 z_4 + z_3 z_4 + z_6 z_4 + z_7 z_4 + z_1 z_5 + z_2 z_5 + z_3 z_5 + z_1 z_6 + z_2 z_6 +$$

$$z_3 z_6 + z_5 z_6 + z_1 z_7 + z_2 z_7 + z_3 z_7 + z_5 z_7$$

$$F = -s z_1 z_3 z_4 - s z_3 z_6 z_4 + s z_2 z_7 z_4 - s z_1 z_3 z_5 - s z_1 z_3 z_6 - s z_1 z_3 z_7$$

$$-s z_1 z_5 z_7 + t z_2 z_7 z_4 - t z_2 z_5 z_6 \quad (21)$$

There is no Euclidean region for this integral. This integral is significantly harder to evaluate then the planar counterpart.

The polynomials U and F can also be determined from the graph theory [2].

II. LEE-POMERANSKY REPRESENTATION

Lee-Pomeransky representation [3, 4] is a modern variant of Feynman parametrization,

$$G[\alpha_1, \dots \alpha_n] = \frac{(-1)^{|\boldsymbol{\alpha}|}\Gamma(D/2)}{\Gamma\big((L+1)D/2 - |\boldsymbol{\alpha}|\big)\Gamma(\alpha_1)\dots\Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n dz_i \, z_1^{\alpha_1-1} \dots z_n^{\alpha_n-1} G^{-D/2} \quad (22)$$

where $G = F + U$, for $\alpha_n > 0$. Note that F and U have different dimensions, and G is apparently a "wrong" expression. How does this formula work?

The Russian trick is to insert a trivial integration into the formula, and rescale $z_i = z_i' s$.

$$\int_0^\infty \prod_{i=1}^n dz_i \int_0^\infty ds \delta(s - \sum_{i=1}^n z_i) \, z_1^{\alpha_1-1} \dots z_n^{\alpha_n-1} G^{-D/2}$$

$$= \int_0^\infty \prod_{i=1}^n dz_i' \int_0^\infty ds \delta(1 - \sum_{i=1}^n z_i') \, z_1'^{\alpha_1-1} \dots z_n'^{\alpha_n-1} s^{|\boldsymbol{\alpha}|-1}(s^L U(z') + s^{L+1} F(z'))^{-D/2}$$

$$= \int_0^\infty \prod_{i=1}^n dz_i' \int_0^\infty ds \delta(1 - \sum_{i=1}^n z_i') \, z_1'^{\alpha_1-1} \dots z_n'^{\alpha_n-1} s^{|\boldsymbol{\alpha}|-1-DL/2}(U(z') + s F(z'))^{-D/2}$$

$$= \frac{\Gamma(|\boldsymbol{\alpha}| - DL/2)\Gamma(-|\boldsymbol{\alpha}| + (D+1)L/2)}{\Gamma(D/2)} \int_0^\infty \prod_{i=1}^n dz_i' \, z_1'^{\alpha_1-1} \dots z_n'^{\alpha_n-1} \delta(1 - \sum_{i=1}^n z_i') \frac{F(z')^{LD/2-|\boldsymbol{\alpha}|}}{U(z')^{(L+1)D/2-|\boldsymbol{\alpha}|}}$$

$$(23)$$

Here we used the Beta function definition. It is clear that the result is consistent with the original Feynman parameterization.

When an index α_i is zero, we need to modify (22) as,

$$\int_0^\infty \frac{dz_i}{\Gamma(\alpha_i)} z_i^{\alpha_i-1} \Big(\dots \Big) \to \Big(\dots \Big)\Big|_{z_i \to 0}. \tag{24}$$

When an index α_i is negative, we need to modify (22) as,

$$\int_0^\infty \frac{dz_i}{\Gamma(\alpha_i)} z_i^{\alpha_i-1} \Big(\dots \Big) \to (-1)^{\alpha_i} \frac{d^{-\alpha_i}}{dz_i^{-\alpha_i}} \Big(\dots \Big)\Big|_{z_i \to 0}. \tag{25}$$

The big advantage of the Lee-Pomeransky representation is that the kernel is simply one factor $G^{-D/2}$. It means the $D-2$ dimensional integral can be reformulated as a combination of D dimensional integrals, because

$$G^{-(D-2)/2} = G^{-D/2}G. \tag{26}$$

This kind of integration of $G^{(\dots)}f(z)$ is a focus of mathematical research. The theory of special functions, the theory of D-modules, the theory of twisted cohomology are all related to this kine of integration. As we will see later, Lee-Pomeransky representation has the special usage for IBPs.

For example, consider the massless box diagram

$$D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2, \quad D_3 = (l_1 - p_1 - p_2)^2, \quad D_4 = (l_1 - p_4)^2, \tag{27}$$

with $p_1 \cdot p_2 = s/2, p_1 \cdot p_4 = t/2, p_1 \cdot p_3 = (-s-t)/2$.

The Lee-Pomeransky polynomial is,

$$G = z_1 + z_2 + z_3 + z_4 - sz_1z_3 - tz_2z_4, \tag{28}$$

Hence

$$G^{(D-2)}[1,1,1,1] = -\frac{2(D-6)}{(D-2)} \Big(G^D[2,1,1,1] + G^D[1,2,1,1] + G^D[1,1,2,1]$$
$$+ G^D[1,1,1,2] \Big) - \frac{2s}{D-2}G^D[2,1,2,1] - \frac{2t}{D-2}G^D[1,2,1,2]. \tag{29}$$

Note that D is *not* an integer and the above dimension-shift identity is regularized by ϵ. The above expression has the correct mass dimension [6].

III. BAIKOV REPRESENTATION

Baikov representation is the duality of Feynman parametrization. This surprisingly simple representation was only discovered in the 1990s. The idea is that instead of integrating over the loop momenta, we integrate over the Lorentz invariant scalar product.

L is the loop order and the l_i's are the loop momenta. We have E independent external vectors that we label as $p_1, ..., p_E$. We assume that the Feynman integrals have been reduced on the integrand level, and set $m = LE + L(L+1)/2$ which equals the number of scalar products in the configuration. The $LE + L(L+1)/2$ products are defined as

$$x_{ij} = l_i \cdot p_j, \quad 1 \le i \le L, \quad 1 \le j \le E, \tag{30}$$

$$y_{ij} = l_i \cdot l_j, \quad 1 \le i \le j \le L \tag{31}$$

The Baikov representation (I) [3, 5] reads,

$$G[n_1, \ldots, n_m] = C_E^L U^{\frac{E-D+1}{2}} \int_\Omega \mathrm{d}\bar{x}\mathrm{d}\bar{y} \; P^{\frac{D-L-E-1}{2}} \frac{1}{D_1^{n_1} \cdots D_m^{n_m}}. \tag{32}$$

Note that D_i's are *linear functions* of the scalar product. So the propagators are linearized. Here, P is the Baikov polynomial, which can be written as a Gram determinant,

$$P = \det G \begin{pmatrix} l_1, & \ldots & l_L, & p_1, & \ldots & p_E \\ l_1, & \ldots & l_L, & p_1, & \ldots & p_E \end{pmatrix}. \tag{33}$$

Moreover, U and C_E^L are the Gram determinants respectively constant factor below:

$$U = \det G \begin{pmatrix} p_1, & \ldots & p_E \\ p_1, & \ldots & p_E \end{pmatrix}, \quad C_E^L = \frac{\pi^{\frac{L-m}{2}}}{\Gamma(\frac{D-E-L+1}{2})\ldots\Gamma(\frac{D-E}{2})}, \tag{34}$$

The Baikov representation (II) [3, 5] reads,

$$G[n_1, \ldots, n_m] = J C_E^L U^{\frac{E-D+1}{2}} \int_\Omega \mathrm{d}^n z \; P^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{n_1} \cdots z_m^{n_m}}. \tag{35}$$

where $z_i = D_m$. Here J, a rational number, is a the Jacobian from (x, y) to z.

Disclaim: We do not treat the overall powers of i in Baikov representation carefully in the expression.

Baikov represent is great for deriving integral relations (IBP, DE and dimension-shift), for the cut analysis. However, usually it is not convenient to evaluate integrals via this expression.

Example: Baikov representation for the one-loop massless box integral.

$$G[\alpha_1, \ldots \alpha_4] = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_4^{\alpha_4}}. \tag{36}$$

We set up a vector basis for the D dimensional spacetime,

$$p_1, p_2, p_4, \omega_1, \ldots, \omega_{D-3} \tag{37}$$

where $p_i \cdot \omega_j = 0$, $\omega_i \cdot \omega_j = -\delta_{ij}$. This is an orthogonal decomposition of the "extra" spacetime. Expand l on this basis,

$$l^\mu = c_1 p_1^\mu + c_2 p_2^\mu + c_3 p_4^\mu + f_1 \omega_1^\mu + \ldots + f_{D-3}\omega_{D-3}^\mu \tag{38}$$

Then we have

$$d^D l = d^3 c \; d^{D-3} f \; \det(p_1, p_2, p_4, \omega_1^\mu, \ldots, \omega_{D-3}^\mu) \tag{39}$$

$$= d^3 c \; d^{D-3} f \det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}^{1/2} \tag{40}$$

$$= dx_1 dx_2 dx_3 \; d^{D-3} f \det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}^{-1/2} \tag{41}$$

where in the second line we used the relation between the Gram determinant and the determinant of the vectors. In the last line we introduced the scalar products,

$$x_1 = l \cdot p_1, \quad x_1 = l \cdot p_2, \quad x_3 = l \cdot p_4, \tag{42}$$

We dropped the power of imaginary "i" in this derivation.

We define $\lambda = f_1^2 + \ldots + f_{D-3}^2$. Then $l^2 = F(c_1, c_2, c_3)^2 - \lambda$, where F is some quadratic function of c's. By simple linear algebra, we see that

$$\lambda = -\frac{\det G \begin{pmatrix} l & p_1, & p_2 & p_4 \\ l & p_1, & p_2 & p_4 \end{pmatrix}}{\det G \begin{pmatrix} p_1, & p_2 & p_4 \\ p_1, & p_2 & p_4 \end{pmatrix}} = -\frac{P}{U}. \tag{43}$$

Therefore we see the measure

$$\int d^D l \to A_{D-4} \int_0^\infty d\lambda \frac{1}{\lambda^{1/2}} \lambda^{\frac{D-4}{2}} \int dx_1 dx_2 dx_3 U^{-\frac{1}{2}}$$
$$= U^{\frac{4-d}{2}} \frac{\pi^{\frac{D-3}{2}}}{\Gamma(\frac{D-3}{2})} \int_0^\infty d\lambda \int dx_1 dx_2 dx_3 P^{\frac{d-5}{2}} \tag{44}$$

The final transformation is that $d(l^2) = df(c_1, c_2, c_3)^2 - d\lambda$. Define $y = l^2$, then

$$\int d^D l \to U^{\frac{4-d}{2}} \frac{\pi^{\frac{D-3}{2}}}{\Gamma(\frac{D-3}{2})} \int dy \int dx_1 dx_2 dx_3 P^{\frac{d-5}{2}} \tag{45}$$

where the integration region is defined by $\lambda > 0$ which is an area with complicated shape in the x_1, x_2, x_3, y space.

What we did above is to derive the Baikov for the one-loop case in a straightforward way. In practice, we need to find the Baikov polynomial P in terms of either the scalar product or the propagators (i.e, Baikov z_i variables).

Note that

$$
G\begin{pmatrix} l & p_1, & p_2 & p_4 \\ l & p_1, & p_2 & p_4 \end{pmatrix} = \begin{pmatrix} y & x_1 & x_2 & x_3 \\ x_1 & 0 & \frac{s}{2} & \frac{1}{2}(-s-t) \\ x_2 & \frac{s}{2} & 0 & \frac{t}{2} \\ x_3 & \frac{1}{2}(-s-t) & \frac{t}{2} & 0 \end{pmatrix}
\tag{46}
$$

- If we use the Baikov representation (32), then

$$
P = \frac{1}{4}\Big(-s^2 ty + s^2 x_1^2 + s^2 x_3^2 + 2s^2 x_1 x_3 - st^2 y + 2st x_1^2 + 2st x_1 x_2 + 2st x_1 x_3
$$
$$
-2st x_2 x_3 + t^2 x_1^2 + t^2 x_2^2 + 2t^2 x_1 x_2 \Big).
\tag{47}
$$

  where $x_1 = l \cdot p_1$, $x_2 = l \cdot p_2$, $x_3 = l \cdot p_4$ and $y = l \cdot l$.

- If we use the Baikov variables z_i's in (35), then

$$
P = \frac{1}{16}\Big(s^2 t^2 - 2s^2 t z_2 - 2s^2 t z_4 + s^2 z_2^2 + s^2 z_4^2 - 2s^2 z_2 z_4 - 2st^2 z_1 - 2st^2 z_3 + 2st z_1 z_2
$$
$$
-4st z_1 z_3 + 2st z_2 z_3 + 2st z_1 z_4 - 4st z_2 z_4 + 2st z_3 z_4 + t^2 z_1^2 + t^2 z_3^2 - 2t^2 z_1 z_3 \Big).
\tag{48}
$$

The constant factor $U = -st(s+t)/4$.

A. First applications of the Baikov representation

- Dimension shift identity. From the Baikov representation, we see that $D + 2$-dimensional representation can be directly rewritten as a linear combination of D-dimensional integrals.

$$
P^{\frac{(D+2)-L-E-1}{2}} \rightarrow P^{\frac{D-L-E-1}{2}} P
\tag{49}
$$

  In practice, usually we prefer using the $D - 2 \rightarrow D$ dimension shift identity from Lee-Pomeransky representation since usually *Lee-Pomeransky polynomial G is simpler than the Baikov polynomial P*. The backward shift $D + 2 \rightarrow D$ identities can then be derived from inverse the $D - 2 \rightarrow D$ shifts.

- Guess integral reduction coefficients. When there is no ISP, the top sector integral reduction coefficients can be "guess" from the residues of Baikov representation. This is the original motive of Baikov representation.

For example, we consider the massless double box $G[1,1,1,1]$ and $G[3,1,1,1]$, it is easy to compute that in the Baikov representation,

$$\text{Res}\Big(G[1,1,1,1])\Big) = U^{\frac{4-D}{2}} \oint_{(0,0,0,0)} \mathrm{d}^4z \; P^{\frac{D-5}{2}} \frac{1}{z_1 \cdots z_4} = \frac{2^{6-d}(s+t)^2(st)^D(-st(s+t))^{-D/2}}{s^3 t^3}$$

(50)

$$\text{Res}\Big(G[3,1,1,1])\Big) = U^{\frac{4-D}{2}} \oint_0 dz_1 \frac{1}{z_1^3} \left(\frac{1}{16}t^2(s-z_1)^2\right)^{\frac{D-5}{2}}$$
$$= \frac{2^{5-D}(D-6)(D-5)(s+t)^2(st)^D(-st(s+t))^{-D/2}}{s^5 t^3}$$

(51)

Then we guess

$$G[3,1,1,1] = \frac{(D-6)(D-5)}{2s^2}G[1,1,1,1] + \text{triangles} + \text{bubbles}.$$

(52)

This guess is consistent with the rigorous IBP computation, which would be introduced in the future.

- Leading Singularity. The great advantage of Baikov representation is that it keeps the pole structures of propagators in Feynman diagrams and the monomials in (35) made the residue computation transparent. Roughly speaking, if we take the residues of all Baikov variables, we get the so-called *leading Singularity*.

For example, for the $4-2\epsilon$ massless one-loop box $G[1,1,1,1]$.

$$JC_E^L U^{\frac{4-D}{2}} \int_\Omega \mathrm{d}^4z \; P^{\frac{D-5}{2}} \frac{1}{z_1 \cdots z_4}$$
$$\to \oint_{(0,0,0,0)} \mathrm{d}^4z \; P^{\frac{D-5}{2}} \frac{1}{z_1 \cdots z_4} = 4/(st)$$

(53)

In the last line, by hand-waving arguments, we set $D \to 4$. Here we do not care about the factor of π or rational constant. $1/(st)$ is the leading singularity of the $4-2\epsilon$ massless one-loop box $G[1,1,1,1]$.

Leading singularity is the most divergent part of the $G[1,1,1,1]$ for singular external kinematics. In the future, we will learn that

$$e^{\epsilon\gamma}(-s)^\epsilon G[1,1,1,1] = \frac{1}{st}\left(\frac{1}{\epsilon^2} - \frac{2\log(t/s)}{\epsilon} - \frac{4\pi^2}{3} + O(\epsilon)\right)$$

(54)

The rational function $1/(st)$, which is divergent if $s \to 0$ or $t \to 0$, matches the leading singularity from the Baikov residue computation.

---

[1] J. M. Henn, Phys. Rev. Lett. **110**, 251601 (2013), 1304.1806.

[2] G. Heinrich, Int. J. Mod. Phys. **A23**, 1457 (2008), 0803.4177.

[3] R. N. Lee and A. A. Pomeransky, JHEP **11**, 165 (2013), 1308.6676.

[4] R. N. Lee, in *Proceedings, 49th Rencontres de Moriond on QCD and High Energy Interactions: La Thuile, Italy, March 22-29, 2014* (2014), pp. 297–300, 1405.5616, URL `https://inspirehep.net/record/1297497/files/arXiv:1405.5616.pdf`.

[5] P. A. Baikov, Phys. Lett. **B385**, 404 (1996), hep-ph/9603267.

[6] This identity is for teaching purpose only. In practice, it is possible to further simplify it by IBPs